An Application of Fractional Differential Equations to Risk Theory

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Abstract

This paper defines a new class of fractional differential operators alongside a family of random variables whose density functions solve fractional differential equations equipped with these operators. These equations can be further used to construct fractional integro-differential equations for the ruin probabilities in collective renewal risk models with inter-arrival time distributions from the aforementioned family. Gamma-time risk models and fractional Poisson risk models are two specific cases among them, whose ruin probabilities have explicit solutions when claim sizes distributions exhibit rational Laplace transforms.

1 Introduction

The concept of first passage time is widely used in financial mathematics and actuarial science. It could model various things, from the the dividend time of a stock to the exercise date of an American put option, or the ruin probability of an insurance company. In this paper we focus on the *ruin time of an insurance business*, namely the first time in which the business surplus (capital) becomes negative. Our analysis is based on solving equations of the probability of ruin as a function of the initial capital (surplus) of the risk process.

Motivated by risk theory applications, we consider a new class of risk processes extending those from [28, 2, 7] into a fractional derivative framework. It has been proved that ruin probabilities are exponential functions when claim sizes follow an exponential distribution, for various inter-arrival time distributions [4]. This paper will derive explicit ruin probabilities in risk models with Erlang-distributed claim sizes and inter-arrival time densities solving fractional differential equations. Gamma-time risk model and fractional Poisson risk model are two particular cases among them. All the results are obtained due to new class of fractional differential operators, which extend those from [5, 34]. These operators generalize the results from [2] to a fractional derivative framework, in which their explicit results concerning ruin probabilities become particular cases. Some existed ruin probability results are retrieved (see Example 4.1 and 4.3 for details), and new results are derived. For instance, in the gamma-time risk model with Erlang(2) distributed claim sizes, the ruin probability has the form

$$A_1 e^{-B_1 u} + A_2 e^{-B_2 u}, \quad u > 0,$$

where A_1,B_1,A_2 and B_2 are constants calculated case by case (see Example 4.2).

The classical collective insurance risk model describes the *surplus* $R(t)$ of an insurance company over time,

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad t > 0 \tag{1}$$

where $u > 0$ is the *initial capital* and $c > 0$ is the *premium rate*. The claims occur randomly. The positive random variable X_i describes the size of the i-th claim, which happened after waiting T_i units of time since the last claim. The process $N(t)$ gives the number of claims that have happened up to time t. In the classical model (1), dating back to [31, 32, 12], all random variables are assumed independent and identically distributed. Moreover, the waiting times are usually assumed to be exponentially distributed, with the resulting counting process $N(t)$ thus being a Poisson process. The *ruin probability* of this compound Poisson risk model, for an initial capital u, is defined as

$$\psi(u) = \mathbb{P}\left(\inf_{t>0} \{R(t) < 0\} \,\middle|\, R(0) = u \right). \tag{2}$$

The net profit condition

$$c\,\mathbb{E}(T_i) > \mathbb{E}(X_i) \tag{3}$$

is imposed to ensure that ruin does not happen with certainty. Various generalizations of the classical risk model (1) have been considered over time. In [39], Sparre Andersen defined the renewal risk model. This model accounts for claim number processes $N(t)$ not necessarily Poisson, but verifying the renewal property. The ruin probabilities $\psi(u)$ in renewal models still solve integral equations, obtained from the renewal property,

$$\psi(u) = \int_0^\infty f_T(t) \left(\int_0^{u+ct} \psi(u + ct - y)\,\mathrm{d}F_X(y) + \int_{u+ct}^\infty \mathrm{d}F_X(y) \right) \mathrm{d}t \tag{4}$$

with the universal boundary condition $\lim_{u\to\infty} \psi(u) = 0$, as in [18]. Here f_T and F_X denote the probability density of the waiting time, and the distribution function of the claim size, respectively. This notation will be used throughout the paper.

There is a large actuarial literature analyzing renewal risk processes. Expressions for the Laplace transform of the ruin probability for risk models with Erlang(2, β) or mixture of 2-exponential waiting times were derived in [13, 14, 15] as solutions of second-order differential equations. [30] calculated the joint and marginal moments of the time of ruin, the surplus before ruin, and the deficit at ruin, whenever the inter-arrival times distributions have rational Laplace-Stieltjes transform. Subsequently, [17] computed the Laplace transform of the non-ruin probability for inter-arrival times distributions exhibiting rational Laplace transforms. [28] used a similar approach as [20] to derive a defective renewal equation for the expected discounted penalty due at ruin in a risk model with Erlang(n) inter-arrival times. Finally, [9] derived linear ordinary differential equations for ruin probabilities in Poisson jump-diffusion processes, with phase-type jumps and obtained explicit results in a few instances. The common thread of these paper consists on deriving the ruin probabilities as solutions of (integro-)differential equations.

In an attempt to develop a general method, [35, 36] introduced two algebraic structures for treating integral operators in conjunction with derivatives, integro-differential operators and integro-differential polynomials. Their method allows the description of the associated differential equations, boundary conditions and solution operators (Green's operator) in a uniform yet formal language. Their algebraic symbolic structures have immediate applications in ruin theory. For instance, as an extension of the Erlang risk model, [2] transformed the integral equation for the expected-discounted-penalty-due-at-ruin function into an integro-differential equation whenever the inter-arrival time distributions have rational Laplace transforms. Rational Laplace transforms densities are equivalent to densities that are solutions of ordinary differential equations with constant coefficients. If the claim size distributions also have rational Laplace transforms, these integro-differential equations can be further reduced to linear boundary value problems. Their symbolic computation approach permits extensions to models with premium dependent on reserves (also discussed in [16] regarding the upper and lower bounds of finite ruin probabilities), the associated boundary problems involving then linear ordinary differential equations with variable coefficients [1]. A similar duality idea has been studied in [25] and the reference therein.

We show that the probability density function of a sum of independent, heterogeneous gamma and Mittag-Leffler random variables satisfies fractional differential equations, which is written in an operator/symbolic form. As an application, we consider a family of risk models with inter-arrival times from this family of distributions, and derive the corresponding fractional integro-differential equations satisfied by the corresponding ruin probabilities. We consider the case of claim sizes described by sums of heterogeneous gamma random variables and show that the corresponding ruin probabilities solve fractional differential equations with constant coefficients. These equations contain both left and right fractional differential operators. We annotate here that these equations can describe other physical phenomena exhibiting anomalous diffusion, as in [22] where the "claim sizes" are height losses of the granular material contained in a silo over time [27]. For other applications, we refer to [19, 23, 29, 41] and the references therein. We also remark that the equations (20) presented in this paper can be seen as generalized cases of the fractional boundary problems treated by [24]. In their analysis, they used critical point theory, for specific fractional differential equations with Dirichlet boundary value conditions.

The gamma-time risk model considered here is the first generalization of the case of Erlang(n)-distributed waiting times considered in [28], to that of waiting times distributed as Gamma(r, λ), r being now any positive real number. This is of significance since, in practice, parameter estimation methods usually yield non-integer-valued shape parameters for the gamma distributions that best fit the available data. It becomes necessary to study the ruin theory related to real-valued gamma-distributed random variables. In this respect, [40] dealt with a special non-integer shape gamma $\Gamma(1/b, 1/b)$, $b > 1$ distributed claims case, and [11] provided three equivalent expressions for ruin probabilities in a Cramér-Lundberg model with gamma distributed claims. Prior to this work, as far as we know, there are no results for non-integer shape gamma-time risk model in the ruin theory literature. The fractional Poisson risk model has been previously treated in [6] and [7] for exponential claim sizes, but here, via this fractional calculus approach, we are able to derive expressions for the ruin probability for a larger class of claim sizes in fractional Poisson models.

The paper is organized as follows. In Section 2 we introduce the concept of fractional integro-differential operators. In Section 3 we present the main result and finally, in Section 4, we perform some illustrative numerical calculations and compare the behavior of the ruin probabilities as a function of the model parameters, for both the gamma-time risk models and fractional Poisson models. Appendix A contains all necessary background on fractional calculus.

2 Fractional Integro-Differential Operators

Let $\mathcal{L}(y)$ denote an n-th degree polynomial $y^n + p_1 y^{n-1} + \cdots + p_{n-1}y + p_n$ and consider the following associated homogeneous ordinary differential equation with constant coefficients

$$\mathcal{L}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)[f](x) = f^{(n)}(x) + p_1 f^{(n-1)}(x) + \cdots + p_{n-1}f'(x) + p_n f(x) = 0. \quad (5)$$

Suppose further that equation (5) can be expressed in the form

$$\bigodot_{j=0}^{m}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \lambda_j\right)^{k_j}[f](x) = 0 \quad (6)$$

for positive real numbers λ_j and integers k_j, $j = 1, \ldots, m$. In (5) and henceforth, $\bigodot$ denotes left-composition of operators, namely

$$\bigodot_{j=1}^{m}\mathcal{L}_j[f] := (\mathcal{L}_m \circ \cdots \circ \mathcal{L}_1)[f].$$

The solution $f(x)$ to (6) is the probability density function of either a sum of Erlang random variables or a mixed Erlang random variable, depending on the boundary conditions (see [2]). We would like generalize equation (6), and characterise its solutions in the case where the exponents k_j are no longer integers.

2.1 Left and Right Fractional Differential Operators

In order to generalize expression (5), it is necessary to explore the world of fractional calculus. Solving fractional differential equations has become an essential issue as fractional-order models appear to be more adequate than previously used integer-order models in various fields. A large host of available analytical

methods for solving fractional order integral and differential equations is discussed in [34], including the Mellin transform method, the power series method, and the symbolic method.
The symbolic method was first introduced in [5] and generalizes the Laplace transform method: it uses a specific expansion (e.g., binomial or geometric) on the differential operator and write it as an infinite sum of fractional derivatives. However, it is always necessary to check the validity of the formal expansion since the interchange of infinite summation and integration requires justification. It is nevertheless a powerful tool for determining the possible form of the solution. Numerous examples of the application of this method to heat and mass transfer problems are discussed by [5].
In this section we define a new family of operators based on the binomial expansion. All of the related definitions and propositions of fractional calculus can be found in Appendix A. The important motivation underlying the following definition comes from realising that for positive integer n and $\alpha \in \mathbb{R}$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + \alpha\right)^n [f](x) = e^{-\alpha x} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(e^{\alpha x} f(x)\right), \tag{7}$$

and similarly for $\left(-\frac{\mathrm{d}}{\mathrm{d}x} + \alpha\right)^n$. We thus define the following operators as the natural generalization in terms of fractional derivatives:

Definition 2.1. Let $r > 0$, $\alpha \in \mathbb{R}$, $a \in [-\infty, \infty)$ and $b \in (-\infty, \infty]$. The left fractional differential operator (LFDO) ${}^{\alpha}_{a}\mathrm{R}^{r}_{x}$ is defined by

$${}^{\alpha}_{a}\mathrm{R}^{r}_{x}\left[f\right](x) := e^{-\alpha x}\, {}_{a}\mathrm{D}^{r}_{x}\left(e^{\alpha x}\, f(x)\right) \tag{8}$$

and the right fractional differential operator (RFDO) ${}^{\alpha}_{x}\mathrm{R}^{r}_{b}$ by

$${}^{\alpha}_{x}\mathrm{R}^{r}_{b}\left[g\right](x) := e^{\alpha x}\, {}^{\mathrm{C}}_{x}\mathrm{D}^{r}_{b}\left(e^{-\alpha x}\, g(x)\right). \tag{9}$$

The domain of definition of ${}^{\alpha}_{a}\mathrm{R}^{r}_{x}$ and ${}^{\alpha}_{x}\mathrm{R}^{r}_{b}$ are those of the left Riemann-Liouville fractional derivatives ${}_{a}\mathrm{D}^{r}_{x}$ and right Caputo fractional derivatives ${}^{\mathrm{C}}_{x}\mathrm{D}^{r}_{b}$ respectively, which are given in Definition A.3 and Definition A.5.

In the case $a = 0$, integration by parts yields the following characterisation of the formal adjoint of ${}^{\alpha}_{0}\mathrm{R}^{r}_{x}$. Along with the integration by parts formula in Proposition A.8, this is the key calculation needed for the proof of our main result.

Proposition 2.1. Let $\alpha \in \mathbb{R}$ and $r > 0$. The formal adjoint with respect to integration by parts of the LFDO ${}^{\alpha}_{0}\mathrm{R}^{r}_{x}$ is the RFDO ${}^{\alpha}_{x}\mathrm{R}^{r}_{\infty}$, namely,

$$\int_0^\infty {}^{\alpha}_{0}\mathrm{R}^{r}_{x}[f](x)\, g(x)\, \mathrm{d}x = \int_0^\infty f(x)\; {}^{\alpha}_{x}\mathrm{R}^{r}_{\infty}[g](x)\, \mathrm{d}x,$$

for appropriate functions f and g (see Proposition A.8).

Note that the LFDO can be used to construct differential equations for probability density functions. Consider a gamma probability density function with shape parameter $r \in \mathbb{R}^+$ and rate parameter $\lambda \in \mathbb{R}^+$, namely

$$f_r(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, \quad x > 0.$$

When r is not an integer, instead of an ordinary differential equation, the gamma density function solves the fractional differential equation

$$
{}_{0}^{\lambda}\mathrm{R}_x^r[f_r](x) = e^{-\lambda x}\ {}_0\mathrm{D}_x^r \left(e^{\lambda x} f_r(x)\right) = 0, \quad x > 0, \tag{10}
$$

with boundary conditions ${}_{0}^{\lambda}\mathrm{R}_x^{r-1}[f_r](0) = \lambda^r$ and ${}_{0}^{\lambda}\mathrm{R}_x^{r-k}[f_r](0) = 0$ for $k = 2, \ldots, \lceil r \rceil$. Another distribution related to the LFDO is the Mittag-Leffler distribution, which is the waiting time distribution in the fractional Poisson process (see in Appendix C). The Mittag-Leffler probability density function with parameter $\mu \in (0,1]$ and $\lambda \in \mathbb{R}^+$ is

$$
f_\mu(x) = \lambda x^{\mu-1} E_{\mu,\mu}(-\lambda x^\mu), \quad t > 0,
$$

and solves the following fractional differential equation

$$
\left({}_{0}^{0}\mathrm{R}_x^\mu + \lambda\right)[f_\mu](x) = ({}_0\mathrm{D}_x^\mu + \lambda)[f_\mu](x) = 0, \quad x > 0, \tag{11}
$$

with the boundary condition ${}_0\mathrm{D}_x^{\mu-1}[f](0) = \lambda$. Here, the function $E_{\mu,\mu}$ is called two-parameter Mittag-Leffler function, which is defined in (32).

2.2 A generalized family of random variables

The next theorem introduces the family of random variables to which the approach presented in this paper applies to. In its full generality, we consider random variables that can be written as finite sums of independent heterogeneous gamma and Mittag-Leffler random variables. At the moment, there is no known explicit formula for the probability density function of such a random variable, but we are always able to express it in a convolution form. Notice that if only gamma random variables with integer shape parameters are involved in the summation, this random variable is the generalized integer gamma distribution (GIG) [10]. We now characterise the fractional boundary value problem satisfied by the density function of such random variables.

Theorem 2.1. Consider a random variable T defined by

$$
T = \sum_{i=1}^{m} Y_i + \sum_{j=1}^{n} Z_j, \tag{12}
$$

in terms of gamma random variables $Y_i \sim \Gamma(r_i, \lambda_{1,i})$ and Mittag-Leffler random variables $Z_j \sim \mathrm{ML}(\mu_j, \lambda_{2,j})$, all independent of each other. Here r_i, $\lambda_{1,i}$, $\lambda_{2,j} \in \mathbb{R}^+$ and $\mu_j \in (0,1]$. Then the density function $f_T^{m,n}(t)$ of T solves the following fractional differential equation

$$
\mathcal{A}_{m,n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)[f_T^{m,n}](t) := \bigodot_{j=1}^{n} \left({}_0\mathrm{D}_t^{\mu_j} + \lambda_{2,j}\right) \bigodot_{i=1}^{m} {}_{0}^{\lambda_{1,i}}\mathrm{R}_t^{r_i}\,[f_T^{m,n}](t) = 0, \tag{13}
$$

with boundary conditions (when $n \neq 0$)

$$
{}_0\mathrm{D}_t^{\mu_1-1} \bigodot_{j=2}^{n} \left({}_0\mathrm{D}_t^{\mu_j} + \lambda_{2,j}\right) \bigodot_{i=1}^{m} {}_{0}^{\lambda_{1,i}}\mathrm{R}_t^{r_i}\ [f_T^{m,n}](t)\big|_{t=0} = \Lambda_{m,n},
$$

$$\text{and}\qquad {}_0\mathrm{D}_t^{\mu_1-k}\bigodot_{j=2}^{n}\left({}_0\mathrm{D}_t^{\mu_j}+\lambda_{2,j}\right)\bigodot_{i=1}^{m}{}^{\lambda_{1,i}}{}_0\mathrm{R}_t^{r_i}\ [f_T^{m,n}](t)|_{t=0}=0,$$

for $k=2,\dots,\left\lceil \sum_{j=1}^{n}\mu_j+\sum_{i=1}^{m}r_i\right\rceil$. Here and subsequently $\Lambda_{m,n}$ denotes

$$\Lambda_{m,n}:=\prod_{i=1}^{m}\lambda_{1,i}^{r_i}\prod_{j=1}^{n}\lambda_{2,j}. \tag{14}$$

Proof. We defer the proof of Theorem 2.1 to Appendix B. □

Remark 2.1. We further assume that all $\lambda_{1,i}$ are different, i.e., $\lambda_{1,i}\neq\lambda_{1,k}$ for all $i\neq k$. In other words, each variable Y_i has the gamma distribution with different rate parameters. The uniqueness of the $\lambda_{1,i}$, rate parameter of the gamma random variable could be realized without any loss of generality. Whenever we have $\lambda_{1,i}=\lambda_{1,k}$, $i\neq k$, we would consider the sum of their corresponding random variables, which is still a gamma random variable.

Remark 2.2. One can show that the boundary conditions in Theorem 2.1 have various equivalent expressions. For any positive integer number $k\leqslant\left\lceil\sum_{i=1}^{m}r_i+\sum_{j=1}^{n}\mu_j\right\rceil$, by choosing non-negative integers $k_{1,i}$ and $k_{2,j}$ such that $\sum_{i=1}^{m}k_{1,i}+\sum_{j=1}^{n}k_{2,j}=k$, we have the boundary conditions of equation (13) as

$$\left(\bigodot_{j=1}^{n}\left({}_0\mathrm{D}_t^{\mu_j-k_{2,j}}+\lambda_{2,j}\cdot{}_0\mathrm{I}_t^{k_{2,j}}\right)\bigodot_{i=1}^{m}{}^{\lambda_{1,i}}{}_0\mathrm{R}_t^{r_i-k_{1,i}}\right)[f_T^{m,n}](t)|_{t=0}=\begin{cases}\Lambda_{m,n}, & k=1\\ 0, & k>1.\end{cases}$$

Remark 2.3. Equation (13) along with its boundary conditions can be regarded as the generalization of a pair of boundary problems discussed in [36]. When the fractional differential algebra is properly defined these fractional-order boundary problems can be factorised and further solved by obtaining their corresponding Green's operators.

The solution to equation (13) depends on the boundary condition. When different boundary conditions are given, we may obtain density functions for other possible random variables. For instance, let us consider the following differential equation with two boundary conditions

$$\begin{cases}\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda\right)^2 f_T^{2,0}(t)=0,\\ \left(\frac{\mathrm{d}}{\mathrm{d}x}+\lambda\right)f_T^{2,0}(t)|_{t=0}=\lambda^2,\\ \lambda f_T^{2,0}(t)|_{t=0}=0.\end{cases}$$

The solution to the above equation is the Elang$(2,\lambda)$ density function $f_T^{2,0}(t)=\lambda^2te^{-\lambda t}$ which belongs to the random variable family considered in the equation

(12). However, the solution to the above equation would become $f_T^{2,0}(t) = \frac{1}{2}\lambda e^{-\lambda t} + \frac{1}{2}\lambda^2 t e^{-\lambda t}$ as long as the boundary condition is changed to

$$\begin{cases} \left(\frac{\mathrm{d}}{\mathrm{d}x} + \lambda\right) f_T^{2,0}(t)|_{t=0} = \frac{1}{2}\lambda^2, \\ \lambda f_T^{2,0}(t)|_{t=0} = \frac{1}{2}\lambda^2. \end{cases}$$

This solution is the density function of mixture exponential and Erlang distribution and the associated distribution does not satisfy the equation (12).

3 Main Results

The LFDO and RFDO give us the ability to study a very general family of distributions that may find applications in various areas, e.g, queuing theory, risk theory and control theory. Although many of the available techniques for the analysis of the associated equations are numerical or asymptotic, the fractional differential approach still offer analytic insights to the related problems. In this section, we aim at accomplishing that with particular problems in the theory of risk. A special family of renewal risk models will be considered, among which the Erlang(n) and fractional Poisson risk models are included. We will show that the ruin probabilities in these models solve fractional integro-differential equations involving our operators.
Before moving to the main result, we introduce a lemma that allows us to change the argument of our operators on a bivariate function under certain circumstances.

Lemma 3.1. For positive real numbers α, r and c, the following identity holds

$$ {}_x^{\alpha}\mathrm{R}_{\infty}^{r}[f(x+cy)](x,y) = c^{-r} \cdot {}_y^{\alpha c}\mathrm{R}_{\infty}^{r}[f(x+cy)](x,y). \tag{15}$$

Proof. We start from the left-hand side of equation (15). By definition we have

$$ {}_x^{\alpha}\mathrm{R}_{\infty}^{r}[f(x+cy)](x,y) = e^{\alpha x}\frac{1}{\Gamma(n-r)}\int_x^{\infty}(t-x)^{n-r-1}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(e^{-\alpha t}f(t+cy)\right)\,\mathrm{d}t.$$

Letting $s = \frac{1}{c}(t-x)+y$ leads to

$$\frac{1}{\Gamma(n-r)}\int_y^{\infty} e^{\alpha c y}(s-y)^{n-r-1}c^{-r}\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left(e^{-\alpha c s}f(cs+x)\right)\,\mathrm{d}s,$$

which is the right-hand side of equation (15). □

Now we are able to generalize the result from [28, 2, 7] to a risk model with inter-arrival times of the form of (12). The main result of this paper is the following:

Theorem 3.1. Consider a renewal risk model

$$R_{m,n}(t) = u + ct - \sum_{i=1}^{N_{m,n}(t)} X_i, \quad t > 0,$$

where the inter-arrival times T_k are assumed to be sum of independent gamma random variables $Y_i \sim \Gamma(r_i, \lambda_{1,i})$ and Mittag-Leffler random variables $Z_j \sim \mathrm{ML}(\mu_j, \lambda_{2,j})$ as in (12). Then the ruin probability $\psi(u)$ under model $R_{m,n}$, satisfies the following fractional integro-differential equation

$$\mathcal{A}^*_{m,n}\left(c\frac{\mathrm{d}}{\mathrm{d}u}\right)[\psi](u) = \Lambda_{m,n}\left(\int_0^u \psi(u-y)\,\mathrm{d}F_X(y) + \int_u^\infty \mathrm{d}F_X(y)\right) \tag{16}$$

with the universal boundary condition $\lim_{u\to\infty}\psi(u) = 0$. Here, the constant $\Lambda_{m,n}$ is given by (14) and $\mathcal{A}^*_{m,n}$ is the formal adjoint of $\mathcal{A}_{m,n}$ (see (13)) and is given by

$$\mathcal{A}^*_{m,n}\left(c\frac{\mathrm{d}}{\mathrm{d}u}\right) := \bigodot_{j=1}^{n}\left(c^{\mu_j}\cdot {}^{\mathrm{C}}_{u}\mathrm{D}^{\mu_j}_{\infty} + \lambda_{2,j}\right)\bigodot_{i=1}^{m}\left(c^{r_i}\cdot {}^{\lambda_{1,i}/c}_{\quad u}\mathrm{R}^{r_i}_{\infty}\right). \tag{17}$$

Proof. For a general renewal risk model, the ruin probability solves the renewal equation (4) (see [18]). Denoting the terms in parentheses of (4) as

$$h(u+ct) = \int_0^{u+ct}\psi(u+ct-y)\,\mathrm{d}F_X(y) + \int_{u+ct}^{\infty}\mathrm{d}F_X(y),$$

we now apply $\mathcal{A}^*_{m,n}\left(c\frac{\mathrm{d}}{\mathrm{d}u}\right)$ on both sides of the renewal equation and use property (15) to obtain

$$\mathcal{A}^*_{m,n}\left(c\frac{\mathrm{d}}{\mathrm{d}u}\right)[\psi](u) = \int_0^\infty f_T^{m,n}(t)\cdot\mathcal{A}^*_{m,n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)[h(u+ct)](u,t)\,\mathrm{d}t.$$

The fractional integration by parts rule (39) is applicable here as

$$\begin{aligned}
&\int_0^\infty f_T^{m,n}(t)\mathcal{A}^*_{m,n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)[h(u+ct)](u,t)\,\mathrm{d}t\\
=&\int_0^\infty \left({}_0\mathrm{D}^{\mu_1}_t + \lambda_{2,1}\right)[f_T^{m,n}](t)\cdot\mathcal{A}^*_{m,n-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)[h(u+ct)](u,t)\,\mathrm{d}t\\
+&\sum_{k=0}^{\lfloor\mu_1\rfloor}\left[(-1)^{\lfloor\mu_1\rfloor+1+k}\cdot{}_0\mathrm{D}^{\mu_1+k-\lfloor\mu_1\rfloor-1}_t[f_T^{m,n}](t)\cdot\mathcal{A}^*_{m,n-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)[h(u+ct)](u,t)\right|_0^\infty.
\end{aligned}$$

The boundary condition term evaluated at $t=0$ could be computed by using the initial value theorem of Laplace transforms,

$${}_0\mathrm{I}^{1-\mu_1}_t[f_T^{m,n}](0) = \lim_{s\to\infty}\left(s^{\mu_1}\cdot\prod_{j=1}^{n}\frac{\lambda_{2,j}}{s^{\mu_j}+\lambda_{2,j}}\cdot\prod_{i=1}^{m}\left(\frac{\lambda_{1,i}}{s+\lambda_{1,i}}\right)^{r_i}\right) = 0.$$

Another boundary condition term evaluated at $t=\infty$ also equals zero due to the fact that the definition of the right Caputo fractional derivative is an integral from t to ∞. Analogously, we are able to move the first n operators $\bigodot_{j=1}^{n}\left({}^{\mathrm{C}}_{t}\mathrm{D}^{\mu_j}_{\infty} + \lambda_{2,j}\right)$ from function h to $f_T^{m,n}$ with all boundary conditions vanishing, which leads to

$$\mathcal{A}^*_{m,n}\left(c\frac{\mathrm{d}}{\mathrm{d}u}\right)[\psi](u) = \int_0^\infty \bigodot_{j=1}^{n}\left({}_0\mathrm{D}^{\mu_j}_t + \lambda_{2,j}\right)[f_T^{m,n}](t)\cdot\bigodot_{i=1}^{m}{}^{\lambda_{1,i}}_{\quad t}\mathrm{R}^{r_i}_{\infty}[h(u+ct)](u,t)\,\mathrm{d}t.$$

Now we use the integration by parts formula in Proposition 2.1 to take the first RFDO ${}^{\lambda_{1,1}}_{\ t}\mathrm{R}^{r_1}_{\infty}$ off of h. Furthermore it can be shown that its adjoint ${}^{\lambda_{1,1}}_{\ 0}\mathrm{R}^{r_1}_{t}$ commutes with $\left({}_0\mathrm{D}^{\mu_j}_t + \lambda_{2,j}\right)$ for all $j = 1, \ldots, n$ on density function $f^{m,n}_T$. We therefore get the right-hand side equal to:

$$
\begin{aligned}
&\int_0^\infty \bigodot_{j=1}^{n} \left({}_0\mathrm{D}^{\mu_j}_t + \lambda_{2,j}\right) {}^{\lambda_{1,1}}_{\ 0}\mathrm{R}^{r_1}_{t}[f^{m,n}_T](t) \cdot \bigodot_{i=2}^{m} {}^{\lambda_{1,i}}_{\ t}\mathrm{R}^{r_i}_{\infty}[h(u+ct)](u,t)\,\mathrm{d}t \\
&+ \sum_{k=0}^{\lfloor r_1 \rfloor} \Bigg[(-1)^{\lfloor r_1 \rfloor + 1 + k} \cdot \bigodot_{i=2}^{m} {}^{\lambda_{1,i}}_{\ t}\mathrm{R}^{r_i}_{\infty}[h(u+ct)](u,t) \\
&\qquad \cdot \bigodot_{j=1}^{n} \left({}_0\mathrm{D}^{\mu_j}_t + \lambda_{2,j}\right) {}^{\lambda_{1,i}}_{\ 0}\mathrm{R}^{r_1+k-\lfloor r_1 \rfloor - 1}_{t}[f^{m,n}_T](t) \Bigg] \Bigg|_0^\infty .
\end{aligned}
$$

The boundary condition at $t = 0$ can be computed by applying the initial value theorem

$$
\begin{aligned}
&\bigodot_{j=1}^{n} \left({}_0\mathrm{D}^{\mu_j}_t + \lambda_{2,j}\right) {}^{\lambda_{1,1}}_{\ 0}\mathrm{R}^{r_1+k-\lfloor r_1 \rfloor - 1}_{t}[f^{m,n}_T](0) \\
&= \prod_{j=1}^{n} \lambda_{2,j} \cdot \lim_{s\to\infty} \Bigg(\frac{\lambda_{1,1}^{r_1} \cdot s}{(s+\lambda_{1,1})^{\lfloor r_1 \rfloor + 1 - k}} \prod_{i=2}^{m} \left(\frac{\lambda_{1,i}}{s+\lambda_{1,i}}\right)^{r_i} \\
&\quad - s \sum_{l=0}^{k-1} (s+\lambda_{1,1})^l \left[{}_0\mathrm{D}^{r_1+k-\lfloor r_1 \rfloor - l - 2}_t \left(e^{\lambda_{1,1}} f^{m,0}_T(t)\right)\right] \Big|_{t=0} \Bigg).
\end{aligned}
$$

We continue to iteratively use the initial value theorem on the terms

$$
s(s+\lambda_{1,1})^l \left[{}_0\mathrm{D}^{r_1+k-\lfloor r_1 \rfloor - l - 2}_t \left(e^{\lambda_{1,1} t} f^{m,0}_T(t)\right)\right] \Big|_{t=0}
$$

until it eventually gives us

$$
s(s+\lambda_{1,1})^{\lfloor r_1 \rfloor - 1} \left[{}_0\mathrm{I}^{\lfloor r_1 \rfloor + 1 - r_1}_t \left(e^{\lambda_{1,1} t} f^{m,0}_T(t)\right)\right] \Big|_{t=0} = s(s+\lambda_{1,1})^{r_1 - 2} \prod_{i=1}^{m} \left(\frac{\lambda_{1,i}}{s}\right)^{r_i},
$$

which tends to zero when $s \to \infty$. The boundary condition term evaluated at $t = \infty$ gives zero since the right Caputo derivatives vanish at infinity. Analogously, we are able to move the rest operators $\bigodot_{i=1}^{m} {}^{\lambda_{1,i}}_{\ t}\mathrm{R}^{r_i}_{\infty}$ from function h to $f^{m,n}_T$ with all boundary conditions vanishing, which leads to

$$
\begin{aligned}
\mathcal{A}^*_{m,n}\left(c\frac{\mathrm{d}}{\mathrm{d}u}\right)[\psi](u) &= \int_0^\infty \mathcal{A}_{m,n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)[f^{m,n}_T](t) \cdot [h(u+ct)](u,t)\,\mathrm{d}t \\
&\quad + \left[[h(u+ct)](u,t) \cdot \mathcal{A}_{m-1,n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)[f^{m,n}_T](t) \right|_{t=0} .
\end{aligned}
$$

Since the time density satisfies equation (13), the integral term of the above equation vanishes. The boundary conditions of $f_T^{m,n}$ ensure that the lower summand is, at $t = 0$,

$$h(u) \bigodot_{j=1}^{n} \left({}_0\mathrm{D}_t^{\mu_j} + \lambda_{2,j}\right) {}^{\lambda_{1,n}}{}_0\mathrm{R}_t^{r_n - 1} \bigodot_{i=1}^{m-1} {}^{\lambda_{1,i}}{}_0\mathrm{R}_t^{r_i} [f_T^{m,n}](0) = \Lambda_{m,n} h(u)$$

This completes the proof. □

Corollary 3.1. The non-ruin probability $\phi(u) = 1 - \psi(u)$ for the risk model in Theorem 3.1 satisfies the following fractional integro-differential equation

$$\mathcal{A}_{m,n}^{*}\left(c\frac{\mathrm{d}}{\mathrm{d}u}\right)[\phi](u) = \Lambda_{m,n}\left(\int_0^u \phi(u-y)\,\mathrm{d}F_X(y)\right) \tag{18}$$

with the universal boundary condition $\lim_{u\to\infty} \phi(u) = 1$ (see (14) and (17) for the definitions of the constant $\Lambda_{m,n}$ and the operator $\mathcal{A}_{m,n}^{*}\left(c\frac{\mathrm{d}}{\mathrm{d}u}\right)$).

Theorem 3.1 characterises a fractional integro-diferential equation satisfied by the ruin probability ψ for a large class of waiting times distributions. Whether or not one can solve for it depends on the particular form of the claim size distribution function F_X.

We now restrict the rest of the analysis to claim sizes X_i distributed as a sum of an arbitrary number of independent Gamma random variables. The next theorem shows that in this case, the whole equation (16) can be written as a boundary value problem with only fractional derivatives. It is important to note that if the claim sizes include any Mittag-Leffler components, as it is the case of T in Theorem 3.1, we would have $\mathbb{E}(X_i) = \infty$ and ruin would happen with probability one since the net profit condition is violated.

Theorem 3.2. Consider the renewal risk model in Theorem 3.1. Assume further that the claim sizes X_i are each distributed as a sum of l independent $\Gamma(s_k, \alpha_k)$ distributed random variables for some $s_k, \alpha_k > 0$, $k = 1, \ldots, l$ i.e.,

$$\mathcal{A}_l\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)[f_X](u) := \bigodot_{k=1}^{l} {}^{\alpha_k}{}_0\mathrm{R}_u^{s_k}[f_X](u) = 0, \tag{19}$$

with certain boundary conditions (see Theorem 2.1). Let $\mathcal{A}_{m,n}^{*}\left(c\frac{\mathrm{d}}{\mathrm{d}u}\right)$ and $\Lambda_{m,n}$ be as defined in (17) and (14) respectively. Then the non-ruin probability $\phi(u)$ satisfies

$$\mathcal{A}_l\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)\mathcal{A}_{m,n}^{*}\left(c\frac{\mathrm{d}}{\mathrm{d}u}\right)[\phi](u) = \Lambda_{m,n}\prod_{k=1}^{l}\alpha_k^{s_k}\cdot\phi(u) \tag{20}$$

with the universal boundary condition $\lim_{u\to\infty} \phi(u) = 1$ and initial-value boundary conditions

$$\left({}^{\alpha_1}{}_0\mathrm{R}_u^{s_1 - k'} \bigodot_{k=2}^{l} {}^{\alpha_k}{}_0\mathrm{R}_u^{s_k} \bigodot_{j=1}^{n}\left(c^{\mu_j}\cdot {}_u^{\mathrm{C}}\mathrm{D}_\infty^{\mu_j} + \lambda_{2,j}\right)\bigodot_{i=1}^{m}\left(c^{r_i}\cdot {}^{\lambda_{1,i}/c}{}_u\mathrm{R}_\infty^{r_i}\right)\right)[\phi](0) = 0, \tag{21}$$

for $k' = 1, \ldots, \left\lceil \sum_{k=1}^{l} s_k \right\rceil - 1$.

Proof. Taking the operator $\mathcal{A}_l\left(\frac{\mathrm{d}}{\mathrm{d}y}\right)$ on two sides of (18) leads to

$$\mathcal{A}_l\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)\mathcal{A}^*_{m,n}\left(c\frac{\mathrm{d}}{\mathrm{d}u}\right)[\phi](u) = \Lambda_{m,n}\cdot\mathcal{A}_l\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)\left(\int_0^u \phi(u-y)f_X(y)\,\mathrm{d}y\right)$$

Recall from Theorem 2.1, we know that the non-ruin probability function $\phi(u)$ is supported on $[0,\infty)$, so the identity

$$\mathcal{A}_l\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)\left(\int_0^u \phi(u-y)f_X(y)\,\mathrm{d}y\right) = \bigodot_{k=1}^{l} {}^{\alpha_k}_{\ 0}\mathrm{R}^{s_k}_u[\phi * f_X](u) = \prod_{k=1}^{l}\alpha^{s_k}\cdot\phi(u)$$

holds in this case, which gives equation (20). For the boundary conditions, we compute

$$\begin{aligned}&\left({}^{\alpha_1}_{\ 0}\mathrm{R}^{s_1-k'}_u \bigodot_{k=2}^{l} {}^{\alpha_k}_{\ 0}\mathrm{R}^{s_k}_u \bigodot_{j=1}^{n}\left(c^{\mu_j}\cdot {}^{\mathrm{C}}_{u}\mathrm{D}^{\mu_j}_{\infty} + \lambda_{2,j}\right)\bigodot_{i=1}^{m}\left(c^{r_i}\cdot {}^{\lambda_{1,i}/c}_{\quad u}\mathrm{R}^{r_i}_{\infty}\right)\right)[\phi](0)\\ =&\Lambda_{m,n}\prod_{k=2}^{l}\alpha_k^{s_k}\cdot {}^{\alpha_1}_{\ 0}\mathrm{R}^{s_1-k'}_u\left(\phi(u) * f_1(u)\,\right)\big|_{u=0},\end{aligned}$$

where f_1 stands for the density function of $\Gamma(s_1,\alpha_1)$. Applying equation (A.6) gives

$$\begin{aligned}&\Lambda_{m,n}\prod_{k=2}^{l}\alpha_k^{s_k}\cdot e^{-\alpha_1 u}\,{}_0\mathrm{D}^{s_1-k'}_u\left[\int_0^u e^{\alpha_1(u-y)}\phi(u-y)\cdot e^{\alpha_1 y}f_1(y)\,\mathrm{d}y\right]\Bigg|_{u=0}\\ =&\Lambda_{m,n}\prod_{k=2}^{l}\alpha_k^{s_k}\left[e^{-\alpha_1 u}\left[e^{\alpha_1 u}\phi(u) * \frac{\alpha_1^{s_1}}{\Gamma(k')}u^{k'-1}\right]\Bigg|_{u=0} + \phi(0)\frac{\alpha_1^{s_1}}{\Gamma(k'+1)}y^{k'}\Bigg|_{y=0}\right],\end{aligned}$$

which equals to zero for $k'=1,\ldots,\left\lceil\sum_{k=1}^{l}s_k\right\rceil-1$. This completes the proof. □

3.1 The characteristic equation method

Our next goal is solving the fractional differential boundary value problem in Theorem 3.2 via a characteristic equation from the ansatz $\phi(u)=e^{-zu}$. The main technical difficulty in the full generality of Theorem 3.2 arises from the fact that the operators in equation (20) combine two different types of differential derivatives: $\mathcal{A}^*_{m,n}\left(c\frac{\mathrm{d}}{\mathrm{d}u}\right)$ is a composition of right Caputo fractional derivatives, while the operators composed in $\mathcal{A}_l\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)$ are LFDOs which are ultimately defined in terms of left Riemann-Liouville fractional derivatives (see (19), (17) and (2.1)). The proposed ansatz is an eigenfunction only for the operators in $\mathcal{A}^*_{m,n}\left(c\frac{\mathrm{d}}{\mathrm{d}u}\right)$ (see Proposition A.9 and Proposition A.10) so we will restrict to the case of $s_k\in\mathbb{N}$, $k=1,\ldots,l$ which simplifies things greatly since

$$\mathcal{A}_l\left(\frac{\mathrm{d}}{\mathrm{d}u}\right) = \bigodot_{k=1}^{l} {}^{\alpha_k}_{\ 0}\mathrm{R}^{s_k}_u = \bigodot_{k=1}^{l}\left(\frac{\mathrm{d}}{\mathrm{d}u}+\alpha_k\right)^{s_k}$$

reduces to a combination of ordinary derivatives.

Note that assuming $s_k \in \mathbb{N}$, $k = 1, \ldots, l$ in (19) is equivalent to assuming that the claim sizes X_i are each distributed as a sum of l independent Erlang random variables. Moreover, under this case, the operator $\mathcal{A}_l\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)\mathcal{A}^*_{m,n}\left(c\frac{\mathrm{d}}{\mathrm{d}u}\right)$ on the left hand side of (20) is a composition of right Caputo fractional derivatives. Furthermore, with the ansatz $\phi(u) = e^{-zu}$, equation (20) yields the following characteristic equation for z:

$$\prod_{k=1}^{l}(-z+\alpha_k)^{s_k} \cdot \prod_{j=1}^{n}(c^{\mu_j} z^{\mu_j} + \lambda_{2,j}) \cdot \prod_{i=1}^{m}(cz+\lambda_{1,i})^{r_i} = \Lambda_{m,n} \cdot \prod_{k=1}^{l} \alpha_k^{s_k}. \tag{22}$$

Note that by the definition of $\Lambda_{m,n}$ in (14), $z = 0$ is always a root of (22). If equation (22) has $N > 0$ additional distinct complex roots with positive real part, say $z_1, \ldots, z_N$, then the non-ruin probability ϕ that solves (20) is

$$\phi(u) = 1 + \sum_{p=1}^{N} K_p e^{-z_p u} \tag{23}$$

The constants K_p, $p = 1, \ldots, N$ are to be determined from the boundary conditions (21) which we now characterise.

Proposition 3.1. Suppose $s_k \in \mathbb{N}$, $k = 1, \ldots, l$, in Theorem 3.2. The number of initial-value boundary conditions of $\phi(u)$ is $N = \sum\limits_{k=1}^{l} s_k$ and they are given explicitly by:

$$\bigodot_{k=1}^{l} {}^{\alpha_k}_{0}\mathrm{R}^{s_{p,k}}_{u} \bigodot_{j=1}^{n}\left(c^{\mu_j} \cdot {}^{\mathrm{C}}_{u}\mathrm{D}^{\mu_j}_{\infty} + \lambda_{2,j}\right) \bigodot_{i=1}^{m}\left(c^{r_i} \cdot {}^{\lambda_{1,i}/c}_{u}\mathrm{R}^{r_i}_{\infty}\right)[\phi](0) = 0,\ p = 1, \ldots, N \tag{24}$$

where the values of $s_{p,k}$ are to be computed as follows: let

$$L(p) = \inf\left\{\ell \in \mathbb{N} : \sum_{k=1}^{\ell} s_k \leqslant p\right\}, \quad p = 1, \ldots, N \tag{25}$$

and define

$$s_{p,k} = \begin{cases} s_k, & \text{if } k < L(p), \\ p - \sum\limits_{i=1}^{L(p)-1} s_i - 1, & \text{if } k = L(p), \\ \vdots & \\ 0, & \text{if } k > L(p). \end{cases}$$

Proof. We consider the p-th boundary condition

$$\begin{aligned} &\bigodot_{k=1}^{l} {}^{\alpha_k}_{0}\mathrm{R}^{s_{p,k}}_{u} \mathcal{A}^*_{m,n}\left(c\frac{\mathrm{d}}{\mathrm{d}u}\right)[\phi](0) \\ =&\Lambda_{m,n} \prod_{k=1}^{L(p)-1} \alpha_k^{s_k}\ {}^{\alpha_{L(p)}}_{0}\mathrm{R}^{s_{p,L(p)}}_{u}\left[\phi * f_{L(p)} * f_{L(p)+}\right](0), \end{aligned}$$

where $f_{L(p)}$ stands for the density function of a $\Gamma\left(s_{L(p)}, \alpha_{L(p)}\right)$ random variable and $f_{L(p)+}$ for the density function of a sum of random varibales with distributions $\Gamma\left(s_k, \alpha_k\right)$, $k = L(p)+1, \ldots, L$. Let $\Phi = \phi * f_{L(p)+}$ and apply Proposition A.6 to compute

$$
\begin{aligned}
{}^{\alpha_{L(p)}}{}_0\mathrm{R}_u^{s_{p,L(p)}} \left[\Phi * f_{L(p)}\right](u) &= \Phi(u)\, {}_0\mathrm{D}_y^{s_{p,L(p)-1}} \left(e^{\alpha_{L(p)} y} f_{L(p)}(y)\right)\Big|_{y=0} \\
&\quad + e^{-\alpha_{L(p)} u} \left[e^{\alpha_{L(p)}(u)} \Phi(u) * {}_0\mathrm{D}_u^{s_{p,L(p)}} e^{\alpha_{L(p)} u} f_{L(p)}(u)\right].
\end{aligned}
$$

Note that $s_{p,L(p)-1} < s_{L(p)}$ and we have

$$
{}^{\alpha_{L(p)}}{}_0\mathrm{R}_u^{s_{p,L(p)}} \left[\Phi * f_{L(p)}\right](0) = \int_0^u \Phi(u-y)\, {}^{\alpha_{L(p)}}{}_0\mathrm{R}_y^{s_{p,L(p)}} f_{L(p)}(y)\, \mathrm{d}y \bigg|_{u=0} = 0.
$$

Since this holds for all $1 \leqslant p \leqslant N$, we complete the proof. $\square$

Substituting the expression (23) for $\phi(u)$ into the boundary conditions (24) yields explicit linear equations for the unknown constants K_p, $p = 1, \ldots, N$. First, denote

$$
\Delta_p := \prod_{j=1}^{n} (c^{\mu_j} z_p^{\mu_j} + \lambda_{2,j}) \prod_{i=1}^{m} (c z_p + \lambda_{1,i})^{r_i}, \quad p = 1, \ldots, N. \tag{26}
$$

Then, the constants K_p, $p = 1, \ldots, N$ in (23) satisfy

$$
\begin{cases}
\Lambda_{m,n} + \sum\limits_{p=1}^{N} \Delta_p K_p = 0 \\
\alpha_1 \Lambda_{m,n} + \Delta \sum\limits_{p=1}^{N} \left(-z_p + \alpha_1\right) K_p = 0 \\
\quad \cdots \\
\alpha_1^{s_1} \Lambda_{m,n} + \sum\limits_{p=1}^{N} \Delta_p \left(-z_p + \alpha_1\right)^{s_1} K_p = 0 \\
\alpha_1^{s_1} \alpha_2 \Lambda_{m,n} + \sum\limits_{p=1}^{N} \Delta_p \left(-z_p + \alpha_1\right)^{s_1} \left(-z_p + \alpha_2\right) K_p = 0 \\
\quad \cdots \\
\alpha_1^{s_1} \alpha_2^{s_2} \Lambda_{m,n} + \sum\limits_{p=1}^{N} \Delta_p \left(-z_p + \alpha_1\right)^{s_1} \left(-z_p + \alpha_2\right)^{s_2} K_p = 0 \\
\quad \cdots \\
\prod\limits_{k=1}^{l-1} \alpha_k^{s_k} \alpha_l^{s_l - 1} \Lambda_{m,n} + \sum\limits_{p=1}^{N} \Delta_p \prod\limits_{k=1}^{l-1} \left(-z_p + \alpha_k\right)^{s_k} \left(-z_p + \alpha_l\right)^{s_l - 1} K_p = 0.
\end{cases} \tag{27}
$$

4 Explicit Expressions for Ruin Probabilities in Gamma-time and Fractional Poisson Risk Models

The class of models considered in Theorem 3.1 is very general. In this section, we thus focus on two specific models which might be of interest to applications, and where explicit forms of ruin (non-ruin) probabilities can be derived.

Remark 4.1. It has been shown [4] that for any renewal risk model, the ruin probability always has an exponential form when the claim distribution is exponential. However, the fractional differential equation approach bridges a solid connection between classical risk model and a class of renewal models, which might be applied in a more sophisticated model.

4.1 Gamma-time Risk Model

A gamma-time risk model, describes the reserve process $R_r(t)$ of an insurance company by replacing the Poisson process $N(t)$ in the classical model (1) with a renewal counting process $N_r(t)$ with $\Gamma(r, \lambda_1)$ distributed waiting times. This is a natural extension of Erlang(n) risk model consiered by [28].
As being a special case of Theorem 3.1, the equation for ruin probability $\psi_r(u)$ in gamma-time risk model is

$$c^r \cdot e^{\frac{\lambda_1}{c}u} \, {}^{\mathrm{C}}_{u}\mathrm{D}^r_{\infty} \left(e^{-\frac{\lambda_1}{c}u}\psi_r(u)\right) = \lambda_1^r \left(\int_0^u \psi_r(u-y)\,\mathrm{d}F_X(y) + \int_u^\infty \mathrm{d}F_X(y)\right).$$

When claim sizes in this model have rational Laplace transforms, one could use the characteristic equation method mentioned in Section 3.1 to derive explicit ruin probabilities.

Example 4.1. In the gamma-time risk model with Gamma(r, λ_1) distributed inter-arrival times and Exp(α) distributed claim sizes, the ruin probability equals to

$$\psi_r(u) = \left(\frac{\lambda_1}{cx_2}\right)^r e^{-\left(x_2 - \frac{\lambda_1}{c}\right)u}, \quad u > 0, \tag{28}$$

where $x_2 > \frac{\lambda_1}{c}$ is the larger root of equation

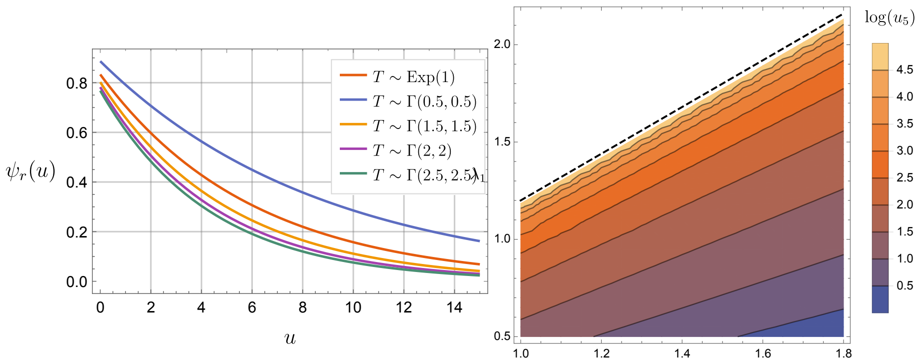
$$c^r x^r \left(x - \left(\frac{\lambda_1}{c} + \alpha\right)\right) + \alpha\lambda_1^r = 0. \tag{29}$$

Remark 4.2. Let $s = x_2 - \frac{\lambda_1}{c}$ in the expression (28), one has

$$\begin{aligned}&(M_X(s)M_T(-cs))^{-1} - 1 = \left(1 - \frac{s}{\alpha}\right)\left(1 + \frac{cs}{\lambda_1}\right)^r - 1\\ =&\frac{c^r}{\lambda_1^r\alpha}\left(\left(\alpha + \frac{\lambda_1}{c} - x_2\right)x_2^r - \frac{\lambda_1^r}{c^r}\right) = \frac{-1}{\lambda_1^r\alpha}\left(c^r x_2^{r+1} - \left(c^{r-1}\lambda_1 + \alpha c^r\right)x_2^r + \alpha\lambda_1^r\right) = 0,\end{aligned}$$

where M_X and M_T are moment generating functions of claim sizes and inter=arrival times. This means that $x_2 - \frac{\lambda_1}{c}$ is the unique positive solution γ of the Lundberg's fundamental equation. This finding coincides with the result from [4] for renewal risk models with exponential claims.

In order to compare the classical and gamma-time risk models, in Figure 1a we show numerically obtained ruin probabilities in the case of Example 4.1 with different combinations of r and λ_1 such that the mean claim inter-arrival time is fixed to $r/\lambda_1 = 1$.



(a) (b)

Figure 1: (a) Ruin probabilities in the case of Example 4.1 for $\lambda_1 = r = 0.5$, 1, 1.5, 2 and 2.5. Claim sizes are taken exponentially distributed with mean $\alpha = 1$ and $c = 1.2$ in order to ensure the net profit condition. (b) Natural log of u_5 (see (30)) for the ruin probability in Example 4.1 with continuously varying parameters r, λ_1. The claim sizes have fixed exponential distribution with mean $\alpha = 1$ and premium rate $c = 1.2$. The dotted line limits the region where the net profit condition $r/\lambda_1 < c$ holds (see (3)).

Note the substantial impact on $\psi_r(u)$ when changing the Poisson assumption ($r = 1$). Ruin probabilities for gamma-time risk model (inter-arrival times $r > 1$) are relatively smaller, and vice versa. The reason is that in this case, the expected inter-arrival time r/λ_1 is fixed whereas the variance of inter-arrival time r/λ_1^2 decreases as r increases, which means that the chance of having a short waiting period between claims will decrease. Since ruin is usually caused by not enough capital accumulating, the model with a larger shape parameter r is more likely to survive. Fugure 1a coincides with the finding from [28], which focuses on Erlang(n) risk models.
In Figure 1b we illustrate the sensitivity to the parameters r, λ_1 of the ruin probability $\psi_r(u)$ in Example 4.1. In order to do this, we define the statistic

$$u_5 := \inf \{u \geq 0 : \psi_r(u) < 0.05\}. \tag{30}$$

Namely, u_5 is the minimum capital needed to achieve a ruin probability of 5%. Note that any combinations of r and λ_1 on or above the dashed line marking the net profit condition, will make the ruin happen for sure. The value of u_5 tends to infinity as the parameters approach the dashed line since the safety loading $\frac{c\,\mathbb{E}(T)}{\mathbb{E}(X)} - 1$ tends to zero. When r takes large enough values or λ_1 take small enough values (in bluer areas), the ruin probability might be less than 5% even with zero initial capital. Note that along contour lines, $\mathrm{d}\lambda_1 \approx \frac{1}{c}\,\mathrm{d}r$, so the sensitivity of the ruin probabilities to its parameters depends almost exclusively on c.
The next example goes a step further and assumes Gamma distributions for both the inter-arrival times and the claim sizes. This case is simple enough that the two positive roots of the characteristic equation can be bounded.

Example 4.2. In the gamma-time risk model with Gamma(r, λ_1) distributed inter-arrival times and Gamma$(2, \alpha)$ distributed claim sizes, the ruin probability equals to

$$\psi_r(u) = \frac{\frac{\lambda_1}{c} - z_3}{z_2 - z_3} \left(\frac{\lambda_1}{cz_2}\right)^r e^{\left(\frac{\lambda_1}{c} - z_2\right)u} + \frac{\frac{\lambda_1}{c} - z_2}{z_3 - z_2} \left(\frac{\lambda_1}{cz_3}\right)^r e^{\left(\frac{\lambda_1}{c} - z_3\right)u}, \quad u > 0,$$

where $z_3 > \frac{\lambda_1}{c} + \alpha > z_2 > \frac{\lambda_1}{c}$ are the two larger roots of the equation

$$c^r z^r \left(z - \left(\frac{\lambda_1}{c} + \alpha\right)\right)^2 - \alpha^2 \lambda_1^r = 0.$$

4.2 Fractional Poisson Risk Model

The fractional (compound) Poisson risk model is a special case of the classic risk model (1) where the counting process is chosen as fractional Poisson process $N_\mu(t)$. Namely the inter-arrival time has Mittag-Leffler distribution $T \sim \mathrm{ML}(\mu, \lambda_2)$ with $\lambda_2 > 0$, $0 < \mu \leqslant 1$. Since when $\mu = 1$, the fractional Poisson process degenerates to the Poisson process, we need the net profit condition to compute the ruin probability. The following examples are under the assumption $0 < \mu < 1$ in the fractional Poisson risk model. Note that in this case $\mathbb{E}T_i = \infty$, so the net profit condition (3) holds whenever $\mathbb{E}X_i < \infty$. It follows from Theorem 3.1 that the ruin probability ψ_μ of a fractional Poisson risk model satisfies the following fractional integro-differential equation

$$c^\mu \, {}_u^{\mathrm{C}}\mathrm{D}_\infty^\mu \psi_\mu(u) + \lambda_2 \psi_\mu(u) = \lambda_2 \left(\int_0^u \psi_\mu(u - y) \, \mathrm{d}F_X(y) + \int_u^\infty \mathrm{d}F_X(y) \right),$$

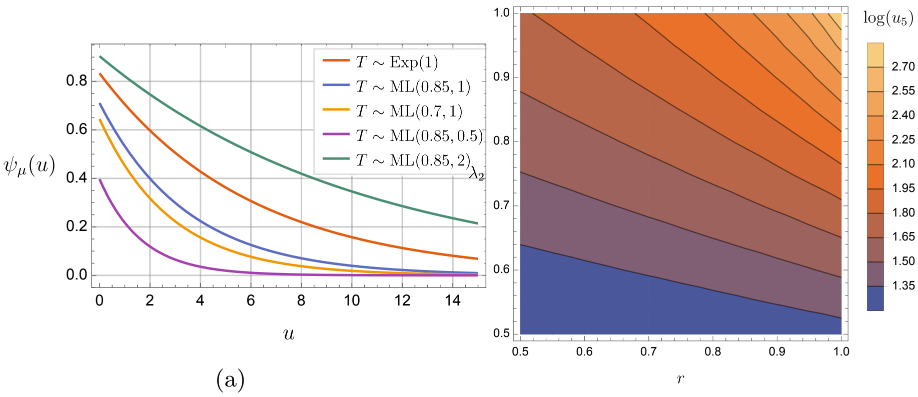
with the universal boundary condition $\lim\limits_{u \to \infty} \psi_\mu(u) = 0$. Explicit expressions for ruin probabilities in fractional Poisson risk model with exponential claims has been derived by [7]. The same result can be obtained via fractional differential equation approach introduced in this paper.

Example 4.3. In the fractional Poisson risk model with $T \sim \mathrm{ML}(\mu, \lambda_2)$ and exponentially distributed claim sizes with parameter α, the ruin probability equals

$$\psi_\mu(u) = \left(1 - \frac{x_2}{\alpha}\right) e^{-x_2 u}, \quad u > 0,$$

where x_2 is the unique positive solution of $c^\mu x - \alpha c^\mu + \lambda_2 x^{1-\mu} = 0$.

Figure 2a shows the ruin probability $\psi_\mu(u)$ for different combinations of the parameters λ_2, μ and fixed exponential claim size distribution.



(a)

(b)

Figure 2: (a) Ruin probabilities in the case of Example 4.3 for different combinations of λ_2, μ. Claim sizes are taken exponentially distributed with mean $\alpha = 1$ and $c = 1.2$. (b) Natural log of u_5 (see (30)) for the ruin probability in Example 4.3 with continuously varying parameters μ, λ_2. The claim sizes have fixed exponential distribution with mean $\alpha = 1$ and premium rate $c = 1.2$.

Note the substantial impact on $\psi_\mu(u)$ when changing the Poisson assumption ($\mu = 1$). Increasing λ_2 or μ increases the chances for ruin to happen. The reason is that, for large enough t, the expected number of jumps before time t in the fractional Poisson process (see equation (41)) is an increasing function of both λ_2 and μ. Moreover, Figure 2b shows the values of natural logarithm of u_5 computed from (30) with ψ_μ as a function of μ and λ_2. Note that the contour lines in this plot are not parallel to each other. As the value of μ decreases, the parameter λ_2 plays a less significant role in the ruin probability function.
Notice that the operator ${}^{\mathrm{C}}_{u}\mathrm{D}^{\mu}_{\infty}$ tends to identity operator when $\mu \to 0+$. Thus, we obtain the following result.

Corollary 4.1. In the fractional Poisson risk model, the ruin probability $\psi_\mu(u)$ converges to a function $\psi_0(u)$, as $\mu \to 0$. Moreover, the function $\psi_0(u)$ satisfies an integral equation

$$(1 + \lambda_2)\,\psi_0(u) = \lambda_2 \int_0^u \psi_0(u - y)\,\mathrm{d}F_X(y) + \lambda_2 \int_u^\infty \mathrm{d}F_X(y), \tag{31}$$

with the universal boundary condition $\lim_{u\to\infty} \psi_0(u) = 0$.

Substituting $u = 0$ into equation (31) gives $\psi_0(0) = \frac{\lambda_2}{\lambda_2+1}$, which only depends on the value of λ_2. Taking Laplace transform both sides with respect to u leads to

$$\hat{\psi}_0(s) = \frac{1 - \hat{f}(s)}{(\lambda_2 + 1)s - \lambda_2 s \hat{f}(s)},$$

which can be explicitly inverted back in some cases.

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A Basic facts from fractional calculus

The fractional calculus is the theory of integrals and derivatives of arbitrary order, which unify and generalize the notions of integer-order differentiation and n-fold integration [34]. The definitions of several special functions, fractional integrals and fractional derivatives used in this paper are listed in this section.

A.1 Mittag-Leffler Function

The Mittag-Leffler function was firstly introduced by [33] as a generalization of the exponential function.

Definition A.1. The two-parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C},\, \Re(\alpha) > 0,\, \Re(\beta) > 0,\, z \in \mathbb{C}. \tag{32}$$

Proposition A.1. The Laplace transform of $z^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm a z^{\alpha})$ is (see [34])

$$\int_0^{\infty} e^{-sz} z^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm a z^{\alpha})\, \mathrm{d}z = \frac{k!\, s^{\alpha-\beta}}{(s^{\alpha} \mp a)^{k+1}}, \quad \Re(s) > |a|^{1/\alpha}. \tag{33}$$

A.2 Fractional Integrals and Derivatives

As per [21], we define and denote:

Definition A.2. The left Riemann-Liouville fractional integral of order $r > 0$ with lower limit $a \in \mathbb{R}$ is defined on locally integrable functions f as

$$ {}_a\mathrm{I}^r_x f(x) = \frac{1}{\Gamma(r)} \int_a^x (x-y)^{r-1} f(y)\, \mathrm{d}y, \quad x > a,$$

and the right Riemann-Liouville fractional integral of order $r > 0$ with upper limit $b \in \mathbb{R}$ is defined as

$$ {}_x\mathrm{I}^r_b f(x) = \frac{1}{\Gamma(r)} \int_x^b (y-x)^{r-1} f(y)\, \mathrm{d}y, \quad x < b.$$

Definition A.3. The left Riemann-Liouville fractional derivative of order $r > 0$ with lower limit a is defined as the integer order derivatives of fractional integrals as follows,

$$ {}_a\mathrm{D}^r_x f(x) = \frac{1}{\Gamma(n-r)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_a^x (x-y)^{n-r-1} f(y)\, \mathrm{d}y, \quad x > a, \tag{34}$$

where $n = \lfloor r \rfloor + 1$, and $\lfloor r \rfloor$ denotes the floor function. Similarly, the right Riemann-Liouville fractional derivative of order $r > 0$ with upper limit b is defined as

$$ {}_x\mathrm{D}^r_b f(x) = (-1)^n \frac{1}{\Gamma(n-r)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_x^b (y-x)^{n-r-1} f(y)\, \mathrm{d}y, \quad x < b. \tag{35}$$

These two operators are well defined on the Lebesgue space $L^{\lceil r \rceil}([a,b])$ (see in [37]). Here, $\lceil r \rceil$ denotes the ceiling function.

Proposition A.2. The Riemann-Liouville fractional derivatives are the left inverse operators of the corresponding fractional integrals (see [42])

$$ {}_a\mathrm{D}^r_x\, {}_a\mathrm{I}^r_x f(x) = f(x) \quad \text{and} \quad {}_x\mathrm{D}^r_b\, {}_x\mathrm{I}^r_b f(x) = f(x), \quad \text{for any } r \in \mathbb{C}. \tag{36}$$

Proposition A.3. The left Riemann-Liouville fractional integrals ${}_a\mathrm{I}^r_x$ and left fractional derivative ${}_a\mathrm{D}^r_x$ of the power function $(x-a)^p$ are (see [34])

$$ {}_a\mathrm{I}^r_x (x-a)^p = \frac{\Gamma(1+p)}{\Gamma(1+p+r)} (x-a)^{p+r} \quad \text{and} \quad {}_a\mathrm{D}^r_x (x-a)^p = \frac{\Gamma(1+p)}{\Gamma(1+p-r)} (x-a)^{p-r}.$$

Proposition A.4. The left fractional derivative ${}_0\mathrm{D}_x^r$ of the two-parameter Mittag-Leffler functions satisfies (see [34])

$$ {}_0\mathrm{D}_x^r \left(x^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\lambda x^\alpha) \right) = x^{\alpha k+\beta-r-1} E_{\alpha,\beta-r}^{(k)}(\lambda x^\alpha). $$

Proposition A.5. The left Riemann-Liouville fractional derivatives ${}_0\mathrm{D}_x^r$ of an integral depending on a parameter $t \in \mathbb{R}$ is given by

$$ {}_0\mathrm{D}_x^r \int_0^x K(x,t)\,\mathrm{d}t = \int_0^\infty {}_t\mathrm{D}_x^r K(x,t)\,\mathrm{d}t + \lim_{t\to x-0} {}_t\mathrm{D}_x^{r-1} K(x,t). $$

Proposition A.6. The left Riemann-Liouville fractional derivatives ${}_0\mathrm{D}_x^r$ of the (positive density) convolution integral equals to

$$ {}_0\mathrm{D}_x^r \left[K * f\right](x) = \left[{}_0\mathrm{D}_t^r K * f\right](t) + \lim_{t\to+0} f(x-t)\ {}_0\mathrm{D}_t^{r-1} K(t). $$

Definition A.4. The Weyl-Liouville fractional derivatives [38, 8] are special cases of the Riemann-Liouville derivatives when a is replaced by $-\infty$ or b is replaced by ∞ in Definition A.3. The right Weyl-Liouville fractional derivative is defined for functions $f \in L^{\lceil r\rceil}([a,b])$ as

$$ {}_x\mathrm{D}_\infty^r f(x) = (-1)^n \frac{1}{\Gamma(n-r)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_x^\infty (y-x)^{n-r-1} f(y)\,\mathrm{d}y, \quad n = \lfloor r\rfloor + 1. $$

Definition A.5. The Caputo fractional derivatives are defined as fractional integrals on integer-order derivatives. The right Caputo fractional derivative is defined on functions $f \in L^{\lceil r\rceil}([a,b])$ as

$$ {}_x^{\mathrm{C}}\mathrm{D}_b^r f(x) = \frac{1}{\Gamma(n-r)} \int_x^b (y-x)^{n-r-1} f^{(n)}(y)\,\mathrm{d}y, \quad x<b, \quad n = \lfloor r\rfloor + 1. \tag{37} $$

Proposition A.7. The Caputo fractional derivatives are the left inverse operators of their corresponding fractional integrals (see [42])

$$ {}_a^{\mathrm{C}}\mathrm{D}_x^r\, {}_a\mathrm{I}_x^r f(x) = f(x) \quad \text{and} \quad {}_x^{\mathrm{C}}\mathrm{D}_b^r\, {}_x\mathrm{I}_b^r f(x) = f(x), \quad \text{for } r \in \mathbb{N} \text{ or } \Re(r) \notin \mathbb{N}. \tag{38} $$

Proposition A.8. The Caputo and left Riemann-Liouville fractional derivatives are related by the following integration by parts formula (see [3])

$$ \int_a^b g(x)\, {}_x^{\mathrm{C}}\mathrm{D}_b^r f(x)\,\mathrm{d}x = \int_a^b f(x)\ {}_a\mathrm{D}_x^r g(x)\,\mathrm{d}x \tag{39} $$

$$ + \sum_{j=0}^{\lfloor r\rfloor} \left[(-1)^{\lfloor r\rfloor+1+j} \left({}_a\mathrm{D}_x^{r+j-\lfloor r\rfloor-1} g(x) \right) \left({}_a\mathrm{D}_x^{\lfloor r\rfloor-j} f(x) \right) \right]_a^b. \tag{40} $$

Proposition A.9. The eigenfunction of left fractional derivative ${}_0\mathrm{D}_x^r$ (or ${}_0^{\mathrm{C}}\mathrm{D}_x^r$) is $x^{1-\alpha} E_{\alpha,\alpha}(\lambda x^\alpha)$ with eigenvalue $\lambda \in \mathbb{R}$ (see [21]).

Proposition A.10. The eigenfunction of right fractional derivative ${}_x\mathrm{D}_\infty^r$ (or ${}_x^{\mathrm{C}}\mathrm{D}_\infty^r$) is $e^{-\lambda x}$ with eigenvalue λ^r, where $\lambda \in \mathbb{R}^+$ (see [42]).

Proposition A.11. The Laplace transform of the left Riemann-Liouville fractional derivative of order $r > 0$ is (see [34])

$$ \mathcal{L}\{{}_0\mathrm{D}_x^r f(x)\}(s) = s^r \hat{f}(s) - \sum_{k=0}^{\lfloor r\rfloor} s^k \left[{}_0\mathrm{D}_x^{r-k-1} f(x)\right]\big|_{x=0} $$

B Proof of Theorem 2.1

Proof. Wwe will use induction on two variables to validate (13) together with the extra statement: for any function g supported on $[0,\infty)$, $\mathcal{A}_{m,n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)[f_T^{m,n} * g](t) = \Lambda_{m,n}\cdot g(t)$. Base step: when $m=1, n=0$ or $m=0, n=1$, from equation (10) and (11) we have $\mathcal{A}_{1,0}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left[f_T^{1,0}\right](t) = 0$ and $\mathcal{A}_{0,1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left[f_T^{0,1}\right](t) = 0$. Furthermore, a simple calculation yields

$$\mathcal{A}_{1,0}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left[f_T^{1,0} * g\right](x) = e^{-\lambda_{1,1}t}\cdot {}_0\mathrm{D}_t^{r_1}\left(e^{\lambda_{1,1}t}\left[f_T^{1,0} * g\right]\right)(t) = \lambda_{1,1}^{r_1}\cdot g(t),$$
$$\mathcal{A}_{0,1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left[f_T^{0,1} * g\right](t) = \left({}_0\mathrm{D}_t^{\mu_1} + \lambda_{2,1}\right)\left[f_T^{0,1} * g\right](t) = \lambda_{2,1}\cdot g(t).$$

Inductive step: for any non-negative m and n, we assume that the statements

$$\mathcal{A}_{m,n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)[f_T^{m,n}](t) = 0, \quad \mathcal{A}_{m,n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)[f_T^{m,n} * g](t) = \Lambda_{m,n}\cdot g(t)$$

hold. We then compute,

$$\mathcal{A}_{m+1,n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left[f_T^{m+1,n}\right](t) = e^{-\lambda_{1,m+1}t}\cdot {}_0\mathrm{D}_t^{r_{m+1}}\left(e^{\lambda_{1,m+1}t}\cdot c_{m,n}\cdot f_T^{1,0}(t)\right) = 0$$
$$\mathcal{A}_{m,n+1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left[f_T^{m,n+1}\right](t) = \left({}_0\mathrm{D}_t^{\mu_{n+1}} + \lambda_{2,n+1}\right)\left(c_{m,n}\cdot f_T^{0,1}(t)\right) = 0,$$
$$\mathcal{A}_{m+1,n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left[f_T^{m+1,n} * g\right](t) = e^{-\lambda_{1,m+1}t}\cdot {}_0\mathrm{D}_t^{r_{m+1}}\left(e^{\lambda_{1,m+1}t}c_{m,n}\cdot f_T^{1,0} * g\right)(t) = c_{m+1,n}\cdot g(t),$$
$$\mathcal{A}_{m,n+1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left[f_T^{m,n+1} * g\right](t) = \left({}_0\mathrm{D}_t^{\mu_{n+1}} + \lambda_{2,n+1}\right)\left[c_{m,n}\cdot f_T^{0,1} * g\right](t) = c_{m,n+1}\cdot g(t),$$

thereby showing $m+1$ and $n+1$ cases are true. To validate the boundary conditions, we compute

$$\begin{aligned}&{}_0\mathrm{D}_t^{\mu_1-k}\bigodot_{j=2}^{n}\left({}_0\mathrm{D}_t^{\mu_j} + \lambda_{2,j}\right)\bigodot_{i=1}^{m}{}^{\lambda_{1,i}}_{\ 0}\mathrm{R}_t^{r_i}\left[f_T^{m,n-1} * f_T^{0,1}\right](0)\\ &= \prod_{i=1}^{m}\lambda_{1,i}^{r_i}\prod_{j=2}^{n}\lambda_{2,j}\cdot {}_0\mathrm{D}_t^{\mu_1-k}\left[f_T^{0,1}\right](0) = \prod_{i=1}^{m}\lambda_{1,i}^{r_i}\prod_{j=2}^{n}\lambda_{2,j}\cdot\lambda_{2,1}t^{k-1}E_{\mu_1,k}(-\lambda_{2,1}t_1^{\mu})\big|_{t=0},\end{aligned}$$

which equals to $\Lambda_{m,n}$ when $k=1$, and 0 for $k>1$. This completes the proof. $\square$

C Review of Fractional Poisson Process

The fractional Poisson process, denoted by $N_\mu(t)$, $t>0$, $\mu\in(0,1]$, is a fractional non-Markovian generalisation of Poisson process $N(t)$, $t>0$. The distribution of fractional Poisson process $P_\mu(n,t) = \mathbb{P}\left(N_\mu(t)=n\right)$ is defined by solving a fractional generalisation of the Kolmogorov-Feller equation [26]

$${}_0\mathrm{D}_t^{\mu}P_\mu(n,t) = \lambda(P_\mu(n-1,t) - P_\mu(n,t)) + \frac{t^{-\mu}}{\Gamma(1-\mu)}\delta_{n,0}, \quad t>0,$$

where λ is the intensity parameter and $\delta_{n,0}$ is the Kronecker symbol. Moreover, [26] showed the inter-arrival times of a fractional Poisson process have probability density function $f_\mu(t) = \lambda t^{\mu-1} E_{\mu,\mu}(-\lambda t^\mu)$, $t > 0$. The Laplace transform of the inter-arrival time density $f_\mu(t)$ is $\mathcal{L}\left\{f_\mu(t); s\right\} = \hat{f}_\mu(s) = \frac{\lambda}{s^\mu + \lambda}$. The mean and variance of $N_\mu(t)$ are

$$\mathbb{E} N_\mu(t) = \frac{\lambda t^\mu}{\Gamma(\mu+1)}, \tag{41}$$

respectively $\mathbb{V}\mathrm{ar}\, N_\mu(t) = \frac{2(\lambda t^\mu)^2}{\Gamma(2\mu+1)} - \frac{(\lambda t^\mu)^2}{(\Gamma(\mu+1))^2} + \frac{\lambda t^\mu}{\Gamma(\mu+1)}$, as in [26].